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Eigenvalue estimates for the Dirac operator with generalized APS boundary condition

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Abstract

Under two boundary conditions, the generalized Atiyah–Patodi–Singer boundary condition and the modified generalized Atiyah–Patodi–Singer boundary condition, we get the lower bounds for the eigenvalues of the fundamental Dirac operator on compact spin manifolds with nonempty boundary.

Keywords: Eigenvalue; Dirac operator; Boundary condition; Ellipticity

1. Introduction

Let M be a compact Riemannian spin manifold without boundary. In 1963, Lichnerowicz [17] proved that any eigenvalue of the Dirac operator satisfies

$$\lambda^2 > \frac{1}{4} \inf_M R,$$

where R is the positive scalar curvature of M. The first sharp estimate for the smallest eigenvalue λ of the Dirac operator D was obtained by Friedrich [7] in 1980. The idea of the proof is based on using a suitably modified Riemannian spin connection. He proved the inequality

$$\lambda^2 \geq \frac{n}{4(n-1)} \inf_M R,$$

on closed manifolds (M^n, g) with the positive scalar curvature R. Equality gives an Einstein metric. In 1986, combining the technique of the modified spin connection with a conformal change of the metric, Hijazi [9] showed, for $n \geq 3$,

$$\lambda^2 \geq \frac{n}{4(n-1)} \mu_1,$$

where μ_1 is the first eigenvalue of the conformal Laplacian given by $L = \frac{4(n-1)}{n-2} \Delta + R$. When the equality holds, the manifold becomes an Einstein manifold.

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Let M be a compact Riemannian spin manifold with nonempty boundary. In this case, the boundary conditions for spinors become crucial to make the Dirac operator elliptic, and, in general, two types of the boundary conditions are considered, i.e. the local boundary condition and the Atiyah–Patodi–Singer (APS) boundary condition (see, e.g. [6]). In 2001, Hijazi et al. [13] proved the generalized version of Friedrich-type inequalities under both the local boundary condition and the Atiyah–Patodi–Singer boundary condition. In 2002, Hijazi et al. [12] investigated the Friedrich-type inequality for eigenvalues of the fundamental Dirac operator under four elliptic boundary conditions and under some curvature assumptions (the non-negative mean curvature). In particular, they introduced a new global boundary condition, namely the modified Atiyah–Patodi–Singer (mAPS) boundary condition.

In the present paper, we study the spectrum of the fundamental Dirac operator on compact Riemannian spin manifolds with nonempty boundary, under the generalized Atiyah–Patodi–Singer (gAPS) boundary condition and the modified generalized Atiyah–Patodi–Singer (mgAPS) boundary condition. Following the terminology of [6] those are obtained by composing a so-called spectral projection with the identity in the first case, and with the zero-order differential operator $\mathrm{Id} + \gamma(e_0)$ in the second case (where $\gamma(e_0)$ is the Clifford product with the unit normal vector field e_0 on the boundary ∂M).

2. Preliminaries

Let (M, g) be an $(n+1)$-dimensional Riemannian spin manifold with nonempty boundary ∂M. We denote by ∇ the Levi-Civitá connection on the tangent bundle TM. For the given structure $Spin\,M$ (and so a corresponding orientation) on manifold M, we denote by $\mathbb{S}$ the associated spinor bundle, which is a complex vector bundle of rank $2^{[\frac{n+1}{2}]}$. Then let

$$\gamma : \mathbb{C}l(M) \longrightarrow End_{\mathbb{C}}(\mathbb{S})$$

be the Clifford multiplication, which is a fibre preserving algebra morphism. It is well known [18] that there exists a natural Hermitian metric $(,)$ on the spinor bundle $\mathbb{S}$ which satisfies the relation

$$(\gamma(X)\varphi, \gamma(X)\psi) = |X|^2(\varphi, \psi), \tag{2.1}$$

for any vector field $X \in \Gamma(TM)$, and for any spinor fields $\varphi, \psi \in \Gamma(\mathbb{S})$. We denote also by ∇ the spinorial Levi-Civitá connection acting on the spinor bundle $\mathbb{S}$. Then the connection ∇ is compatible with the Hermitian metric $(,)$ and Clifford multiplication "γ". Recall that the fundamental Dirac operator D is the first-order elliptic differential operator acting on the spinor bundle $\mathbb{S}$, which is locally given by

$$D = \sum_{i=0}^{n} \gamma(e_i)\nabla_i, \tag{2.2}$$

where $\{e_0, e_1, \ldots, e_n\}$ is a local orthonormal frame of TM.

Since the normal bundle to the boundary hypersurface is trivial, the Riemannian manifold ∂M is also a spin manifold and so we have a corresponding spinor bundle $\mathbb{S}\partial M$, a spinorial Levi-Civitá connection $\nabla^{\partial M}$ and an intrinsic Dirac operator $D^{\partial M} = \gamma^{\partial M}(e_i)\nabla_i^{\partial M}$. Then a simple calculation yields the spinorial Gauss formula

$$\nabla_i\varphi = \nabla_i^{\partial M}\varphi + \frac{1}{2}h_{ij}\gamma(e_j)\gamma(e_0)\varphi,$$

for $1 \le i, j \le n$, $\varphi \in \Gamma(\mathbb{S}|_{\partial M})$; e_0 is a unit normal vector field compatible with the induced orientation, and h_{ij} is the second fundamental form of the boundary hypersurface.

It is easy to check (see [4,11,12,14]) that the restriction of the spinor bundle $\mathbb{S}$ of M to its boundary is related to the intrinsic Hermitian spinor bundle $\mathbb{S}\partial M$ by

$$\mathcal{S} := \mathbb{S}|_{\partial M} \cong \begin{cases} \mathbb{S}\partial M & \text{if } n \text{ is even} \\ \mathbb{S}\partial M \oplus \mathbb{S}\partial M & \text{if } n \text{ is odd.} \end{cases}$$

For any spinor field $\psi \in \Gamma(\mathcal{S})$ on the boundary ∂M, define on the restricted spinor bundle $\mathcal{S}$ the Clifford multiplication $\gamma^{\mathcal{S}}$ and the spinorial connection $\nabla^{\mathcal{S}}$ by

$$\gamma^{\mathcal{S}}(e_i)\psi := \gamma(e_i)\gamma(e_0)\psi \tag{2.3}$$

$$\nabla_i^{\mathcal{S}}\psi := \nabla_i\psi - \frac{1}{2}h_{ij}\gamma^{\mathcal{S}}(e_j)\psi = \nabla_i\psi - \frac{1}{2}h_{ij}\gamma(e_j)\gamma(e_0)\psi. \tag{2.4}$$

One can easily find that $\nabla^{\mathcal{S}}$ is compatible with the Clifford multiplication $\gamma^{\mathcal{S}}$, the induced Hermitian inner product $(,)$ from M. Moreover, $\nabla^{\mathcal{S}}$ satisfies the following additional identity

$$\nabla_i^{\mathcal{S}}(\gamma(e_0)\psi) = \gamma(e_0)\nabla_i^{\mathcal{S}}\psi. \tag{2.5}$$

As a consequence, the boundary Dirac operator $\mathcal{D}$ associated with the connection $\nabla^{\mathcal{S}}$ and the Clifford multiplication $\gamma^{\mathcal{S}}$ is locally given by

$$\mathcal{D}\psi = \gamma^{\mathcal{S}}(e_i)\nabla_{e_i}^{\mathcal{S}}\psi = \gamma(e_i)\gamma(e_0)\nabla_{e_i}\psi + \frac{H}{2}\psi, \tag{2.6}$$

where $H = \sum h_{ii}$ is the mean curvature of ∂M. In fact, $\gamma(e_0)(\mathcal{D} - \frac{1}{2}H) = \gamma(e_i)\nabla_i$ is the hypersurface Dirac operator defined by Witten [19] to prove the positive energy conjecture in general relativity.

Now by (2.5), we have the supersymmetry property

$$\mathcal{D}\gamma(e_0) = -\gamma(e_0)\mathcal{D}. \tag{2.7}$$

Hence, when ∂M is compact, the spectrum of $\mathcal{D}$ is symmetric with respect to zero and coincides with the spectrum of $D^{\partial M}$ for n even and with (Spec $D^{\partial M}$) $\cup$ ($-$Spec $D^{\partial M}$) for n odd.

From [11] or [12], one gets the integral form of the Schrödinger–Lichnerowicz formula for a compact Riemannian spin manifold with compact boundary:

$$\int_{\partial M}(\varphi, \mathcal{D}\varphi) - \frac{1}{2}\int_{\partial M} H|\varphi|^2 = \int_M |\nabla\varphi|^2 + \frac{R}{4}|\varphi|^2 - |D\varphi|^2. \tag{2.8}$$

By (2.7), $\mathcal{D}$ is self-adjoint with respect to the induced Hermitian metric $(,)$ on $\mathcal{S}$. Therefore, $\mathcal{D}$ has a discrete spectrum contained in $\mathbb{R}$ numbered like

$$\cdots \le \lambda_{-j} \le \cdots \lambda_{-1} < 0 \le \lambda_0 \le \lambda_1 \le \cdots \le \lambda_j \le \cdots$$

and one can find an orthonormal basis $\{\varphi_j\}_{j\in\mathbb{Z}}$ of $L^2(\partial M; \mathcal{S})$ consisting of eigenfunctions of $\mathcal{D}$ (i.e. $\mathcal{D}\varphi_j = \lambda_j\varphi_j$, $j \in \mathbb{Z}$) (see e.g. Lemma 1.6.3 in [8]). Such a system $\{\lambda_j; \varphi_j\}_{j\in\mathbb{Z}}$ is called a *spectral decomposition* of $L^2(\partial M; \mathcal{S})$ generated by $\mathcal{D}$, or, in short, a *spectral resolution* of $\mathcal{D}$. According to Definition 14.1 in [5], we have

Definition 2.1. For the self-adjoint Dirac operator $\mathcal{D}$ and for any real b we shall denote by

$$P_{\ge b} : L^2(\partial M; \mathcal{S}) \longrightarrow L^2(\partial M; \mathcal{S})$$

the *spectral projection*, that is, the orthogonal projection of $L^2(\partial M; \mathcal{S})$ onto the subspace spanned by $\{\varphi_j | \lambda_j \ge b\}$, where $\{\lambda_j; \varphi_j\}_{j\in\mathbb{Z}}$ is a spectral resolution of $\mathcal{D}$.

Definition 2.2. For $\varphi \in \Gamma(\mathcal{S})$ and $b \le 0$, the projective boundary conditions

$$P_{\ge b}\varphi = 0,$$
$$P^m_{\ge b}\varphi = P_{\ge b}(\mathrm{Id} + \gamma(e_0))\varphi = 0$$

are called the generalized Atiyah–Patodi–Singer (gAPS) boundary condition and the modified generalized Atiyah–Patodi–Singer (mgAPS) boundary condition respectively.

Remark 2.1. We shall adopt the notation $P_{<b} = Id - P_{\ge b}$; $P_{[b,-b]} = P_{\ge b} - P_{>-b}$ for $b < 0$ or $P_{[-b,b]} = P_{\ge -b} - P_{>b}$ for $b > 0$. It is well known that the spectral projection $P_{\ge b}$ is a pseudo-differential operator of order zero (see e.g. Proposition 14.2 in [5]).

Remark 2.2. If $b = 0$, the gAPS boundary condition is exactly the APS boundary condition (see Atiyah et al. [1–3]).

In the case of a closed manifold, the fundamental Dirac operator is a formally self-adjoint operator and so its spectrum is discrete and real, while in the case of a manifold with nonempty boundary, we have the following formula:

$$\int_M (D\varphi, \psi) - \int_M (\varphi, D\psi) = -\int_{\partial M} (\gamma(e_0)\varphi, \psi), \tag{2.9}$$

for $\varphi, \psi \in \Gamma(\partial M)$; e_0 is the inner unit normal vector field along the boundary. In the following, we will show ellipticity and self-adjointness for the Dirac operator under both the gAPS boundary condition and the mgAPS boundary condition.

On the one hand, if $P_{\geq b}\varphi = 0$, $P_{\geq b}\psi = 0$, for $\varphi, \psi \in \Gamma(\mathcal{S})$ and $b \leq 0$, then $P_{<b}(\gamma(e_0)\varphi) = \gamma(e_0)P_{>-b}$. Therefore

$$\int_{\partial M} (\gamma(e_0)\varphi, \psi) = 0. \tag{2.10}$$

On the other hand, for $\varphi, \psi \in \Gamma(\mathcal{S})$ and $b \leq 0$, under the mgAPS boundary conditions, i.e.

$$P_{\geq b}(\mathrm{Id} + \gamma(e_0))\varphi = 0 \quad \text{and} \quad P_{\geq b}(\mathrm{Id} + \gamma(e_0))\psi = 0,$$

a simple calculation implies that (2.10) also holds.

These facts imply that the fundamental Dirac operator D is self-adjoint under the gAPS boundary condition and the mgAPS boundary condition and hence it has real and discrete eigenvalues.

The ellipticity of both gAPS and mgAPS boundary conditions for the fundamental Dirac operator D can be proved in the same way as [15].

Lemma 2.1. *Under the gAPS boundary condition $P_{\geq b}\varphi = 0$, for $\varphi \in \Gamma(\mathcal{S})$, the inequality*

$$\int_{\partial M} (\varphi, \mathcal{D}\varphi) < b \int_{\partial M} |\varphi|^2$$

holds for $b \leq 0$.

Proof. Let $\{\lambda_j; \varphi_j\}_{j\in\mathbb{Z}}$ be a spectral resolution of the hypersurface Dirac operator $\mathcal{D}$. Then any $\varphi \in \Gamma(\mathcal{S})$ can be expressed as follows:

$$\varphi = \sum_j c_j \varphi_j,$$

where $c_j = \int_{\partial M} (\varphi, \varphi_j)$. Then we have

$$\int_{\partial M} (\varphi, \mathcal{D}\varphi) = \sum_{\lambda_j < b} \lambda_j |c_j|^2 < b \sum_{\lambda_j < b} |c_j|^2 = b \int_{\partial M} |\varphi|^2. \quad \square$$

3. Lower bounds for the eigenvalues

In this section, we adapt the arguments used in [15] and [13] to the case of compact Riemannian spin manifolds with nonempty boundary. We use the integral identity (2.8) together with an appropriate modification of the Levi-Cività connection to obtain the eigenvalue estimates.

Theorem 3.1. *Let M^n be a compact Riemannian spin manifold of dimension $n \geq 2$, with nonempty boundary ∂M, and let λ be any eigenvalue of D under the gAPS boundary condition $P_{\geq b}\varphi = 0$, for $\varphi \in \Gamma(\mathcal{S})$ and $b \leq 0$. If there exist real functions a, u on M such that*

$$b + a\, du(e_0) \leq \frac{1}{2} H$$

on ∂M, then

$$\lambda^2 > \frac{n}{4(n-1)} \sup_{a,u} \inf_M R_{a,u}, \tag{3.1}$$

where

$$R_{a,u} := R - 4a\Delta u + 4\nabla a \cdot \nabla u - 4\left(1 - \frac{1}{n}\right) a^2 |du|^2. \tag{3.2}$$

Δ is the positive scalar Laplacian, and R is the scalar curvature of M.

Proof. For any real functions a and u, define the modified spinorial connection (see [13,15]) on $\Gamma(\mathbb{S})$ by

$$\nabla_i^{a,u} = \nabla_i + a\nabla_i u + \frac{a}{n}\nabla_j u\gamma(e_i)\gamma(e_j) + \frac{\lambda}{n}\gamma(e_i). \tag{3.3}$$

The following identity was proved in [13]:

$$\int_M |\nabla^{a,u}\varphi|^2 = \int_M \left[\left(1 - \frac{1}{n}\right)\lambda^2 - \frac{R_{a,u}}{4}\right]|\varphi|^2 + \int_{\partial M}(\varphi, \mathcal{D}\varphi) + \int_{\partial M}\left[a\, du(e_0) - \frac{H}{2}\right]|\varphi|^2. \tag{3.4}$$

Considering Lemma 2.1 and $b + a\, du(e_0) \leq \frac{1}{2}H$, we obtain

$$\lambda^2 \geq \frac{n}{4(n-1)} \sup_{a,u} \inf_M R_{a,u}. \tag{3.5}$$

If the equality in (3.5) holds, then by Lemma 3 in [13], we have $a = 0$ or $u =$ constant and $b \leq \frac{1}{2}H$. Since φ is a non-degenerated killing spinor, $|\varphi|^2$ is a nonzero constant. Under the gAPS boundary condition $P_{\geq b}\varphi = 0$, we have

$$0 = \int_{\partial M}(\varphi, \mathcal{D}\varphi) - \int_{\partial M}\frac{1}{2}H|\varphi|^2 < \int_{\partial M}(b - \frac{1}{2}H)|\varphi|^2 \leq 0.$$

This is a contradiction. Thus the inequality (3.1) holds. □

Remark 3.1. If $b = 0$, the gAPS boundary condition becomes the APS boundary condition and then the theorem is exactly Theorem 3.1 in [13].

Now we make use of the energy–momentum tensor, introduced in [10] and used in [13–16], to get lower bounds for the eigenvalues of D. For any spinor field $\varphi \in \Gamma(\mathbb{S})$, we define the associated energy–momentum 2-tensor Q_φ on the complement of its zero set by

$$Q_{\varphi,ij} = \frac{1}{2}\Re\left(\gamma(e_i)\nabla_j\varphi + \gamma(e_j)\nabla_i\varphi, \varphi/|\varphi|^2\right).$$

Obviously, $Q_{\varphi,ij}$ is a symmetric tensor. If φ is the eigenspinor of the Dirac operator D, the tensor Q_φ is well-defined in the sense of distribution.

Theorem 3.2. *Let M^n be a compact Riemannian spin manifold of dimension $n \geq 2$, whose boundary ∂M is nonempty, and let λ be any eigenvalue of D under the gAPS boundary condition $P_{\geq b}\varphi = 0$, for $b \leq 0$, $\varphi \in \Gamma(\mathcal{S})$. If there exist real functions a, u on M such that*

$$b + adu(e_0) \leq H/2$$

on ∂M, where H is the mean curvature of ∂M, then

$$\lambda^2 \geq \sup_{a,u} \inf_M \left(\frac{1}{4}R_{a,u} + |Q_\varphi|^2\right),$$

where $R_{a,u}$ is given in (3.2).

Proof. For any real functions a and u, define the modified spinorial connection (see [13,15]) on $\Gamma(\mathbb{S})$ by

$$\nabla_i^{Q,a,u} = \nabla_i + a\nabla_i u + \frac{a}{n}\nabla_j u\gamma(e_i)\gamma(e_j) + Q_{\varphi,ij}\gamma(e_j). \tag{3.6}$$

One can easily compute

$$\int_M |\nabla^{Q,a,u}\varphi|^2 = \int_M \left[\lambda^2 - \left(\frac{R_{a,u}}{4} + |Q_\varphi|^2\right)\right]|\varphi|^2 + \int_{\partial M}(\varphi, \mathcal{D}\varphi) + \int_{\partial M}[adu(e_0) - H/2]\,|\varphi|^2. \tag{3.7}$$

Considering the boundary conditions, we immediately arrive at the asserted formula. □

Remark 3.2. By the Cauchy–Schwarz inequality and $tr\,Q_\varphi = \lambda$, we have

$$|Q_\varphi|^2 = \sum_{i,j} |Q_{\varphi,ij}|^2 \geq \sum\nolimits_i |Q_{\varphi,ii}|^2 \geq \frac{1}{n}(tr\,Q_\varphi)^2 = \frac{1}{n}\lambda^2.$$

Therefore, one gets the inequality (3.5).

Remark 3.3. If $b = 0$, the above theorem becomes Theorem 5 in [13]. Under the gAPS boundary condition, taking $a = 0$ or $u =$ constant in (3.6) and assuming $H \geq 0$, then one gets the following inequality [10]:

$$\lambda^2 \geq \inf_M \left(\frac{R}{4} + |Q_\varphi|^2\right).$$

In the following, we state the eigenvalue estimates for the Dirac operator under the mgAPS boundary condition.

Theorem 3.3. *Let M^n be a compact Riemannian spin manifold of dimension $n \geq 2$, whose boundary ∂M is nonempty, and let λ be any eigenvalue of D under the mgAPS boundary condition $P^m_{\geq b} = P_{\geq b}(\mathrm{Id} + \gamma(e_0)) = 0$, for $b \leq 0$. If there exist real functions a, u on M such that*

$$a\,du(e_0) \leq \frac{1}{2}H \tag{3.8}$$

on ∂M, then

$$\lambda^2 \geq \frac{n}{4(n-1)} \sup_{a,u}\ \inf_M R_{a,u}, \tag{3.9}$$

where $R_{a,u}$ is given in (3.2). Moreover, the equality holds if and only if M carries a nontrivial real killing spinor field with a killing constant $-\frac{\lambda}{n} < \frac{b}{n-1}$ and the boundary ∂M is minimal.

Proof. The proof is in the spirit of [12]. Assuming

$$P^m_{\geq b}\varphi = P_{\geq b}(\varphi + \gamma(e_0)\varphi) = 0 \quad \text{and} \quad P^m_{\geq b}\psi = P_{\geq b}(\psi + \gamma(e_0)\psi) = 0,$$

for $\varphi, \psi \in \Gamma(\mathcal{S})$ and $b \leq 0$, then we have

$$P_{>-b}\psi + \gamma(e_0)P_{<b}\psi = 0 \quad \text{and} \quad P_{[b,-b]}\psi + \gamma(e_0)P_{[b,-b]}\psi = 0,$$

i.e.

$$P^m_{\geq b}\psi = P_{>-b}\psi + \gamma(e_0)P_{<b}\psi = 0 \quad \text{and} \quad P_{[b,-b]}\psi = 0. \tag{3.10}$$

These imply

$$P^m_{\geq b}\varphi = P_{>-b}(\varphi + \gamma(e_0)\varphi) = 0,$$
$$P^m_{\geq b}\psi = P_{>-b}(\psi + \gamma(e_0)\psi) = 0.$$

Then one gets

$$\gamma(e_0)\psi - \psi = P_{>-b}(\gamma(e_0)\psi - \psi). \tag{3.11}$$

The relation (2.7) implies that

$$(\mathcal{D}\psi, \psi) = \frac{1}{2}(\mathcal{D}(\psi + \gamma(e_0)\psi), \psi - \gamma(e_0)\psi).$$

The combination $P^m_{\geq b}\psi = P^m_{>-b}\psi = P_{>-b}(\psi + \gamma(e_0)\psi) = 0$ with (3.11) yields

$$\int_{\partial M} (\mathcal{D}(\psi + \gamma(e_0)\psi), \psi - \gamma(e_0)\psi) = 0.$$

This implies

$$\int_{\partial M} (\mathcal{D}\psi, \psi) = 0. \tag{3.12}$$

From (3.3), (3.4) and (3.12), one can obtain

$$\int_M |\nabla^{a,u}\varphi|^2 = \int_M \left[\left(1-\frac{1}{n}\right)\lambda^2 - \frac{R_{a,u}}{4}\right]|\varphi|^2 + \int_{\partial M}\left[a\, du(e_0) - \frac{H}{2}\right]|\varphi|^2. \tag{3.13}$$

Then the inequality (3.9) holds.

If the equality occurs, we deduce that

$$\nabla^{a,u}\varphi = 0 \quad \text{and} \quad \frac{1}{2}H = adu(e_0).$$

By Lemma 3 in [13], we get $a = 0$ or $u =$ constant. Therefore

$$H = 0 \quad \text{and} \quad \nabla_i\varphi = -\frac{\lambda}{n}\gamma(e_i)\varphi.$$

From the supersymmetry property (2.7), we get

$$\mathcal{D}(\varphi + \gamma(e_0)\varphi) = -\frac{n-1}{n}\lambda(\varphi + \gamma(e_0)\varphi).$$

Since $P^m_{\geq b}\varphi = 0$, we deduce that $-\frac{n-1}{n}\lambda < b$.

Conversely, if M is a compact Riemannian spin manifold with minimal boundary ∂M and a nontrivial killing spinor φ with a real killing constant $-\lambda/n < \frac{b}{n-1}$, then we have $D\varphi = \lambda\varphi$. Moreover, from the fact that $\nabla_{e_0}\varphi = -\frac{\lambda}{n}\gamma(e_0)\varphi$, we infer that the restriction of φ to the boundary satisfies

$$\mathcal{D}\varphi = -\frac{n-1}{n}\gamma(e_0)\varphi.$$

Finally, we have

$$\mathcal{D}(\varphi + \gamma(e_0)\varphi) = -\frac{n-1}{n}\lambda(\varphi + \gamma(e_0)\varphi).$$

Since the spinor field $\varphi + \gamma(e_0)\varphi$ is an eigenspinor of $\mathcal{D}$ with a eigenvalue $-\frac{(n-1)}{n}\lambda < b$, one can infer that $P^m_{\geq b}\varphi = 0$.
□

Remark 3.4. Under the mgAPS boundary condition, taking $a = 0$ or $u =$ constant in (3.8) and (3.13), one gets Friedrich's inequality

$$\lambda^2 \geq \frac{n}{4(n-1)} \inf_M R.$$

In particular, if we take $b = 0$, the above theorem is exactly Theorem 5 in [12].

Using the energy–momentum tensor, (3.7) and (3.12), we get

$$\int_M |\nabla^{Q,a,u}\varphi|^2 = \int_M \left[\lambda^2 - \left(\frac{R_{a,u}}{4} + |Q_\varphi|^2\right)\right]|\varphi|^2 + \int_{\partial M}[adu(e_0) - H/2]\,|\varphi|^2.$$

Thus we obtain the following theorem:

Theorem 3.4. *Let M^n be a compact Riemannian spin manifold of dimension $n \geq 2$, with nonempty boundary ∂M, and let λ be any eigenvalue of D under the mgAPS boundary condition $P^m_{\geq b}\varphi = P_{\geq b}(\varphi + \gamma(e_0)\varphi) = 0$, for $b \leq 0$, $\varphi \in \Gamma(\mathcal{S})$. If there exist real functions a, u on M such that*

$$a\, du(e_0) \leq \frac{1}{2}H$$

on ∂M, then

$$\lambda^2 \geq \sup_{a,u}\ \inf_M \left(\frac{1}{4}R_{a,u} + |Q_\varphi|^2\right), \tag{3.14}$$

where $R_{a,u}$ is given in (3.2).

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